Chapter 12

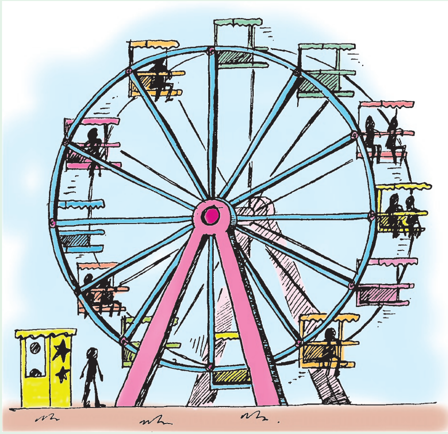
Trigonometric functions

Syllabus reference: 3.2, 3.4, 3.5

Contents:

OPENING PROBLEM

A Ferris wheel rotates at a constant speed. The wheel's radius is 10 m and the bottom of the wheel is 2 m above ground level. From a point in front of the wheel, Andrew is watching a green light on the perimeter of the wheel. Andrew notices that the green light moves in a circle. He estimates how high the light is above ground level at two second intervals and draws a scatter diagram of his results.



Things to think about:

a What does his scatter diagram look like?

b What function can be used to model the data?

c How could this function be used to find:

  i the light's position at any point in time

  ii the times when the light is at its maximum and minimum heights?

d What part of the function would indicate the time interval over which one complete cycle occurs?

Click on the icon to visit a simulation of the Ferris wheel. You will be able to view the light from:

- in front of the wheel
- a side-on position
- above the wheel.

DEMO

You can then observe the graph of the green light's position as the wheel rotates at a constant rate.

A PERIODIC BEHAVIOUR

Periodic phenomena occur all the time in the physical world. For example, in:

- seasonal variations in our climate
- variations in average maximum and minimum monthly temperatures
- the number of daylight hours at a particular location
- tidal variations in the depth of water in a harbour
- the phases of the moon
- animal populations.

These phenomena illustrate variable behaviour which is repeated over time. The repetition may be called **periodic**, **oscillatory**, or **cyclic** in different situations.

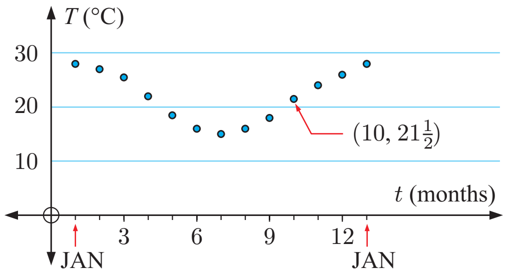
In this chapter we will see how trigonometric functions can be used to model periodic phenomena.

OBSERVING PERIODIC BEHAVIOUR

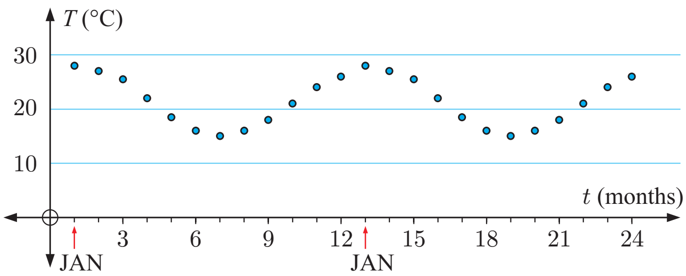
The table below shows the mean monthly maximum temperature for Cape Town, South Africa.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Temp. T (°C)	28	27	$25\frac{1}{2}$	22	$18\frac{1}{2}$	16	15	16	18	$21\frac{1}{2}$	24	26

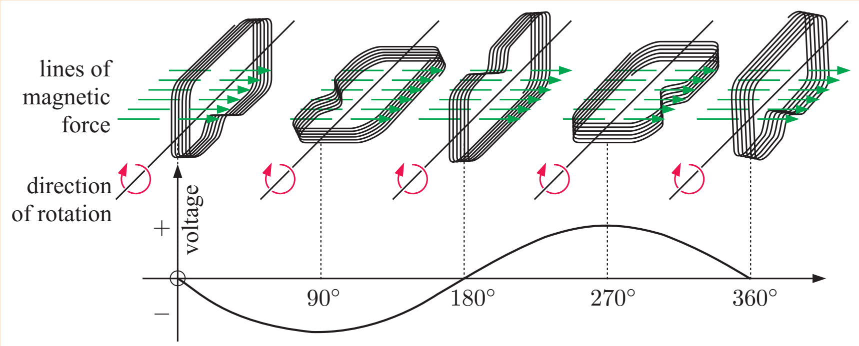
On the scatter diagram alongside we plot the temperature T on the vertical axis. We assign January as $t = 1$ month, February as $t = 2$ months, and so on for the 12 months of the year.



The temperature shows a variation from an average of 28°C in January through a range of values across the months. The cycle will approximately repeat itself for each subsequent 12 month period. By the end of the chapter we will be able to establish a **periodic function** which approximately fits this set of points.



HISTORICAL NOTE



In 1831 **Michael Faraday** discovered that an electric current was generated by rotating a coil of wire in a magnetic field. The electric current produced showed a voltage which varied between positive and negative values as the coil rotated through 360°.

Graphs with this basic shape, where the cycle is repeated over and over, are called **sine waves**.

GATHERING PERIODIC DATA

Data on a number of periodic phenomena can be found online or in other publications. For example:

- Maximum and minimum monthly temperatures can be found at www.weatherbase.com
- Tidal details can be obtained from daily newspapers or internet sites such as http://tidesandcurrents.noaa.gov or http://www.bom.gov.au/oceanography

TERMINOLOGY USED TO DESCRIBE PERIODICITY

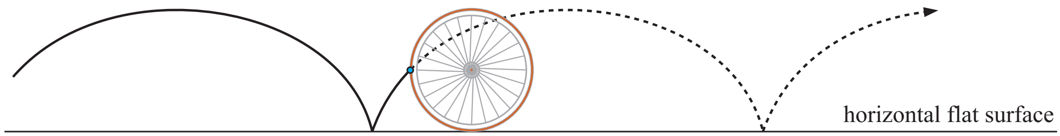
A **periodic function** is one which repeats itself over and over in a horizontal direction, in intervals of the same length.

The **period** of a periodic function is the length of one repetition or cycle.

$f(x)$ is a periodic function with period $p \Leftrightarrow \boldsymbol{f(x+p) = f(x)}$ for all x, and p is the smallest positive value for this to be true.

A **cycloid** is an example of a periodic function. It is the curve traced out by a point on a circle as the circle rolls across a flat surface in a straight line.

DEMO

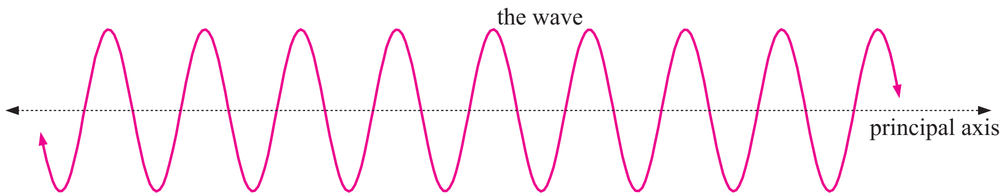


Use a **graphing package** to examine the function $f(x) = x - [x]$ where $[x]$ is "the largest integer less than or equal to x".
Is $f(x)$ periodic? What is its period?

GRAPHING PACKAGE

WAVES

In this course we are mainly concerned with periodic phenomena which show a wave pattern:

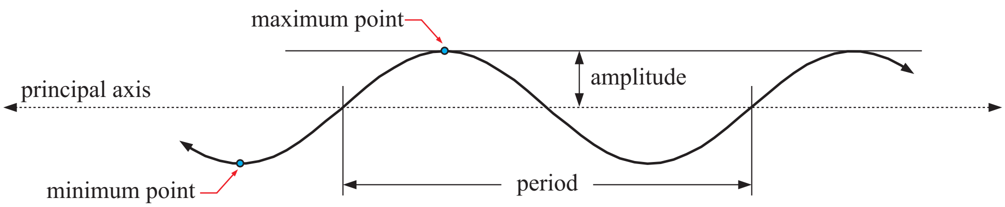


The wave oscillates about a horizontal line called the **principal axis** or **mean line** which has equation $\boldsymbol{y = \dfrac{\textbf{max} + \textbf{min}}{2}}$.

A **maximum point** occurs at the top of a crest, and a **minimum point** at the bottom of a trough.

The **amplitude** is the distance between a maximum (or minimum) point and the principal axis.

$$\textbf{amplitude} = \frac{\textbf{max} - \textbf{min}}{2}$$



EXERCISE 12A

1 Which of these graphs show periodic behaviour?

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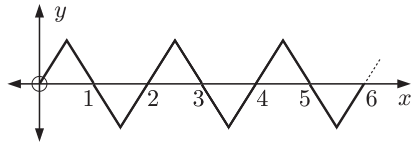

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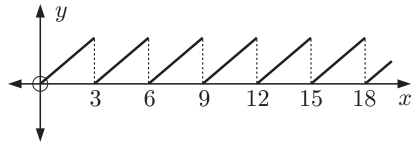
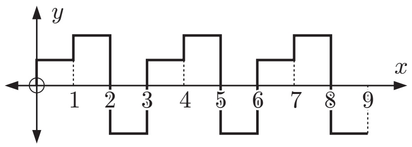


c


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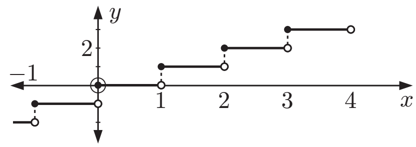

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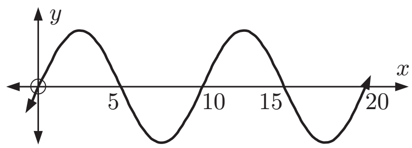
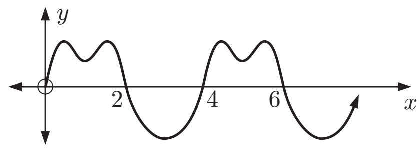


f


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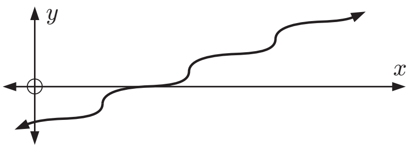
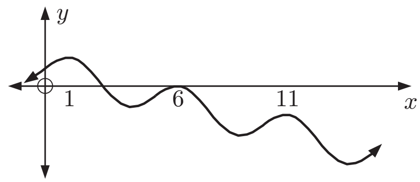


h


2 The following tabled values show the height above the ground of a point on a bicycle wheel as the bicycle is wheeled along a flat surface.

Distance travelled (cm)	0	20	40	60	80	100	120	140	160	180	200
Height above ground (cm)	0	6	23	42	57	64	59	43	23	7	1

Distance travelled (cm)	220	240	260	280	300	320	340	360	380	400
Height above ground (cm)	5	27	40	55	63	60	44	24	9	3

a Plot the graph of height against distance.

b Is it reasonable to fit a curve to this data, or should we leave it as discrete points?

c Is the data periodic? If so, estimate:

- **i** the equation of the principal axis
- **ii** the maximum value
- **iii** the period
- **iv** the amplitude.

3 Draw a scatter diagram for each set of data below. Is there any evidence to suggest the data is periodic?

a

x	0	1	2	3	4	5	6	7	8	9	10	11	12
y	0	1	1.4	1	0	−1	−1.4	−1	0	1	1.4	1	0

b

x	0	2	3	4	5	6	7	8	9	10	12
y	0	4.7	3.4	1.7	2.1	5.2	8.9	10.9	10.2	8.4	10.4

GRAPHICS CALCULATOR INSTRUCTIONS

THEORY OF KNOWLEDGE

In mathematics we clearly define terms so there is no misunderstanding of their exact meaning.

We can understand the need for specific definitions by considering integers and rational numbers:

- 2 is an integer, and is also a rational number since $2 = \frac{4}{2}$.
- $\frac{4}{2}$ is a rational number, and is also an integer since $\frac{4}{2} = 2$.
- $\frac{4}{3}$ is a rational number, but is *not* an integer.

Symbols are frequently used in mathematics to take the place of phrases. For example:

- $=$ is read as "is equal to"
- $\sum$ is read as "the sum of all"
- $\in$ is read as "is an element of" or "is in".

1 Is mathematics a language?

2 Why is it important that mathematicians use the same notation?

3 Does a mathematical argument need to read like a good piece of English?

The word *similar* is used in mathematics to describe two figures which are in proportion. This is different from how *similar* is used in everyday speech.

Likewise the words *function*, *domain*, *range*, *period*, and *wave* all have different or more specific mathematical meanings.

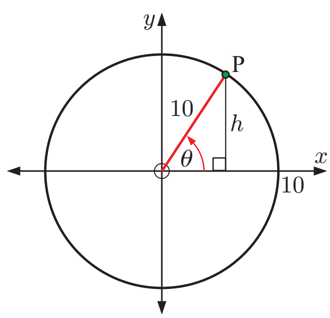
4 What is the difference between *equal*, *equivalent*, and *the same*?

5 Are there any words which we use only in mathematics? What does this tell us about the nature of mathematics and the world around us?

B THE SINE FUNCTION

In previous studies of trigonometry we have only considered static situations where an angle is fixed. However, when an object moves around a circle, the situation is dynamic. The angle θ between the radius [OP] and the positive x-axis continually changes with time.

Consider again the **Opening Problem** in which a Ferris wheel of radius 10 m revolves at constant speed. We let P represent the green light on the wheel.



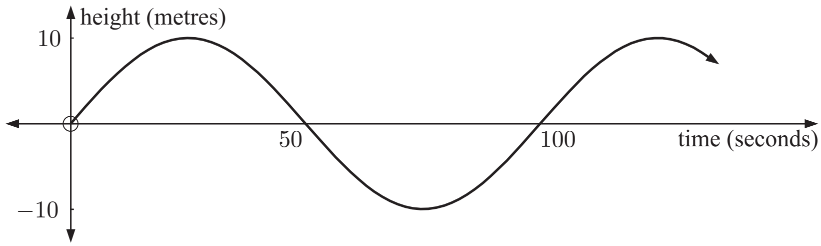
The height of P relative to the x-axis can be determined using right angled triangle trigonometry.

$$\sin\theta = \frac{h}{10}, \quad \text{so} \quad h = 10\sin\theta.$$

As time goes by, θ changes and so does h.

So, we can write h as a function of θ, or alternatively we can write h as a function of time t.

Suppose the Ferris wheel observed by Andrew takes 100 seconds for a full revolution. The graph below shows the height of the light above or below the principal axis against the time in seconds.

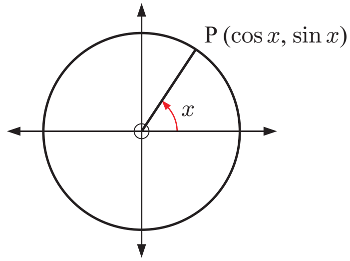


DEMO

We observe that the amplitude is 10 metres and the period is 100 seconds.

THE BASIC SINE CURVE

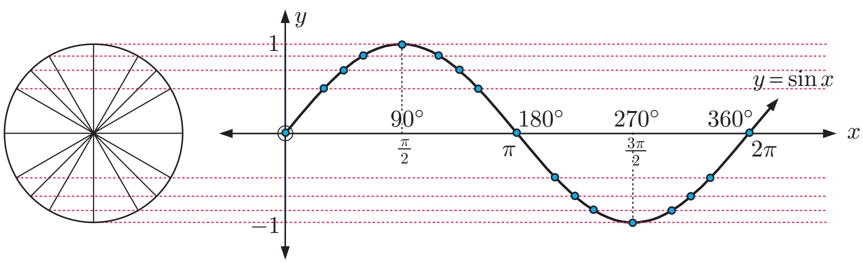
Suppose point P moves around the unit circle so the angle [OP] makes with the positive horizontal axis is x. In this case P has coordinates $(\cos x, \sin x)$.



If we project the values of $\sin x$ from the unit circle to a set of axes alongside, we can obtain the graph of $y = \sin x$.

Note carefully that x on the unit circle diagram is an *angle*, and becomes the horizontal coordinate of the sine function.

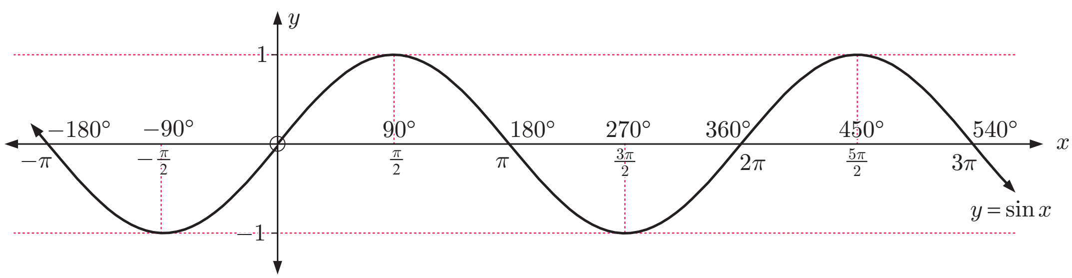
Unless indicated otherwise, you should assume that x is measured in radians. Degrees are only included on this graph for the sake of completeness.



Click on the icon to generate the sine function for yourself.

SINE FUNCTION

You should observe that the sine function can be continued beyond $0 \leqslant x \leqslant 2\pi$ in either direction.



The unit circle repeats itself after one full revolution, so its *period* is 2π.

The *maximum* value is 1 and the *minimum* is -1, as $-1 \leqslant y \leqslant 1$ on the unit circle.

The *amplitude* is 1.

TRANSFORMATIONS OF THE SINE CURVE

In the investigations that follow, we will consider applying transformations to the sine curve $y = \sin x$. Using the transformations we learnt in **Chapter 5**, we can generate the curve for the general sine function $y = a\sin(b(x - c)) + d$.

INVESTIGATION 1 — THE FAMILY $y = a\sin x,\ a \neq 0$

Click on the icon to explore the family $y = a\sin x$, $a \neq 0$.

DYNAMIC SINE FUNCTION

Notice that x is measured in radians.

What to do:

1 Use the slider to vary the value of a. Observe the changes to the graph of the function.

2 Use the software to help complete the table:

a	*Function*	*Maximum*	*Minimum*	*Period*	*Amplitude*
1	$y = \sin x$	1	-1	2π	1
2	$y = 2\sin x$				
0.5	$y = 0.5\sin x$				
-1	$y = -\sin x$				
a	$y = a\sin x$				

3 How does a affect the function $y = a\sin x$?

4 State the amplitude of:

a $y = 3\sin x$ **b** $y = \sqrt{7}\sin x$ **c** $y = -2\sin x$

INVESTIGATION 2 — THE FAMILY $y = \sin bx,\ b > 0$

Click on the icon to explore the family $y = \sin bx$, $b > 0$.

DYNAMIC SINE FUNCTION

What to do:

1 Use the slider to vary the value of b. Observe the changes to the graph of the function.

2 Use the software to help complete the table:

b	*Function*	*Maximum*	*Minimum*	*Period*	*Amplitude*
1	$y = \sin x$	1	-1	2π	1
2	$y = \sin 2x$				
$\frac{1}{2}$	$y = \sin(\frac{1}{2}x)$				
b	$y = \sin bx$				

3 How does b affect the function $y = \sin bx$?

4 State the period of:

a $y = \sin 3x$ **b** $y = \sin(\frac{1}{3}x)$ **c** $y = \sin(1.2x)$ **d** $y = \sin bx$

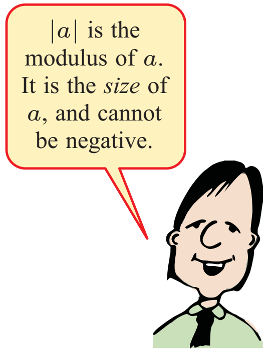
From the previous **Investigations** you should have found:

Family $\boldsymbol{y = a\sin x,\quad a \neq 0}$

- a affects the amplitude of the graph; amplitude $= |a|$
- The graph is a vertical stretch of $y = \sin x$ with scale factor $|a|$.
- If $a < 0$, the graph of $y = \sin x$ is also reflected in the x-axis.

Family $\boldsymbol{y = \sin bx,\quad b > 0}$

- The graph is a horizontal stretch of $y = \sin x$ with scale factor $\frac{1}{b}$.
- period $= \dfrac{2\pi}{b}$



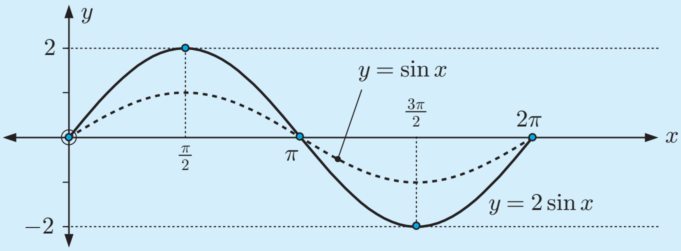
Example 1

Self Tutor

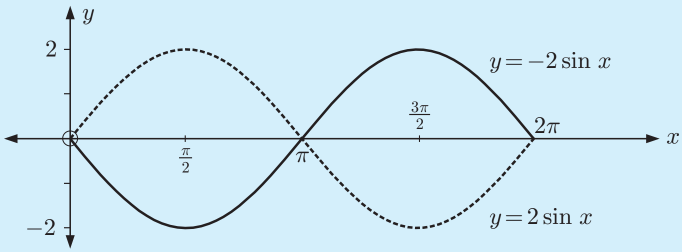
Without using technology, sketch the graphs of:

a $y = 2\sin x$ **b** $y = -2\sin x$ for $0 \leqslant x \leqslant 2\pi$.

a This is a vertical stretch of $y = \sin x$ with scale factor 2.
The amplitude is 2 and the period is 2π.



b The amplitude is 2 and the period is 2π. It is the reflection of $y = 2\sin x$ in the x-axis.



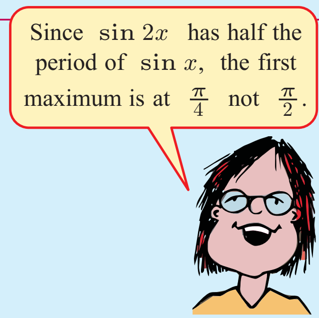
Example 2

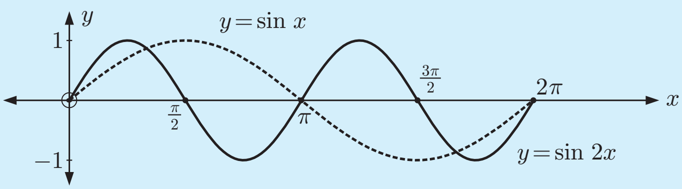
Self Tutor

Without using technology, sketch the graph of $y = \sin 2x$ for $0 \leqslant x \leqslant 2\pi$.

This is a horizontal stretch of $y = \sin x$ with scale factor $\frac{1}{2}$.

The period is $\frac{2\pi}{2} = \pi$, so the maximum values are π units apart.





EXERCISE 12B.1

1 Without using technology, sketch the graphs of the following for $0 \leqslant x \leqslant 2\pi$:

a $y = 3\sin x$ **b** $y = -3\sin x$ **c** $y = \frac{3}{2}\sin x$ **d** $y = -\frac{3}{2}\sin x$

2 Without using technology, sketch the graphs of the following for $0 \leqslant x \leqslant 3\pi$:

a $y = \sin 3x$ **b** $y = \sin\left(\frac{x}{2}\right)$ **c** $y = \sin(-2x)$

3 State the period of:

a $y = \sin 4x$ **b** $y = \sin(-4x)$ **c** $y = \sin\left(\frac{x}{3}\right)$ **d** $y = \sin(0.6x)$

4 Find b given that the function $y = \sin bx$, $b > 0$ has period:

a 5π **b** $\frac{2\pi}{3}$ **c** 12π **d** 4 **e** 100

5 Use a graphics calculator or graphing package to help you graph for $0° \leqslant x \leqslant 720°$:

a $y = 2\sin x + \sin 2x$ **b** $y = \sin x + \sin 2x + \sin 3x$ **c** $y = \dfrac{1}{\sin x}$

6 **a** Use a graphing package or graphics calculator to graph for $-4\pi \leqslant x \leqslant 4\pi$:

GRAPHING PACKAGE

i $f(x) = \sin x + \dfrac{\sin 3x}{3} + \dfrac{\sin 5x}{5}$

ii $f(x) = \sin x + \dfrac{\sin 3x}{3} + \dfrac{\sin 5x}{5} + \dfrac{\sin 7x}{7} + \dfrac{\sin 9x}{9} + \dfrac{\sin 11x}{11}$

b Predict the graph of $f(x) = \sin x + \dfrac{\sin 3x}{3} + \dfrac{\sin 5x}{5} + \dfrac{\sin 7x}{7} + ... + \dfrac{\sin 1001x}{1001}$

INVESTIGATION 3 THE FAMILIES $y = \sin(x - c)$ AND $y = \sin x + d$

Click on the icon to explore the families $y = \sin(x - c)$ and $y = \sin x + d$.

DYNAMIC SINE FUNCTION

What to do:

1 Use the slider to vary the value of c. Observe the changes to the graph of the function, and complete the table:

c	*Function*	*Maximum*	*Minimum*	*Period*	*Amplitude*
0	$y = \sin x$	1	-1	2π	1
-2	$y = \sin(x - 2)$				
2	$y = \sin(x + 2)$				
$-\frac{\pi}{3}$	$y = \sin(x - \frac{\pi}{3})$				
c	$y = \sin(x - c)$				

2 What transformation moves $y = \sin x$ to $y = \sin(x - c)$?

3 Return the value of c to zero, and now vary the value of d. Observe the changes to the graph of the function, and complete the table:

d	*Function*	*Maximum*	*Minimum*	*Period*	*Amplitude*
0	$y = \sin x$	1	-1	2π	1
3	$y = \sin x + 3$				
-2	$y = \sin x - 2$				
d	$y = \sin x + d$				

4 What transformation moves $y = \sin x$ to $y = \sin x + d$?

5 What transformation moves $y = \sin x$ to $y = \sin(x - c) + d$?

From **Investigation 3** we observe that:

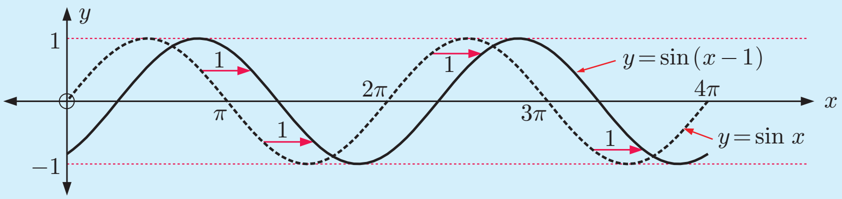
- $y = \sin(x - c)$ is a **horizontal translation** of $y = \sin x$ through c units.
- $y = \sin x + d$ is a **vertical translation** of $y = \sin x$ through d units.
- $y = \sin(x - c) + d$ is a **translation** of $y = \sin x$ through vector $\binom{c}{d}$.

Example 3 **Self Tutor**

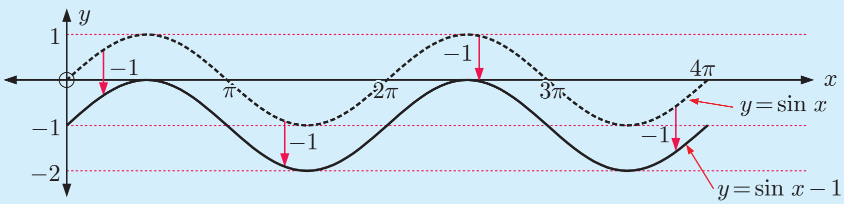
On the same set of axes graph for $0 \leqslant x \leqslant 4\pi$:

a $y = \sin x$ and $y = \sin(x - 1)$ **b** $y = \sin x$ and $y = \sin x - 1$

a This is a horizontal translation of $y = \sin x$ to the right by 1 unit.



b This is a vertical translation of $y = \sin x$ downwards by 1 unit.



THE GENERAL SINE FUNCTION

The **general sine function** is

$$\boldsymbol{y = a\sin(b(x - c)) + d} \quad \text{where } b > 0.$$

- a affects **amplitude**
- b affects **period**
- c affects **horizontal translation**
- d affects **vertical translation**

The **principal axis** of the general sine function is $\boldsymbol{y = d}$.

The **period** of the general sine function is $\boldsymbol{\dfrac{2\pi}{b}}$.

The **amplitude** of the general sine function is $|\boldsymbol{a}|$.

For example, $y = 2\sin(3(x - \frac{\pi}{4})) + 1$ is a translation of $y = 2\sin 3x$ with translation vector $\begin{pmatrix} \frac{\pi}{4} \\ 1 \end{pmatrix}$.

Starting with $y = \sin x$ we would:

- double the amplitude to produce $y = 2\sin x$, then
- divide the period by 3 to produce $y = 2\sin 3x$, then
- translate by $\begin{pmatrix} \frac{\pi}{4} \\ 1 \end{pmatrix}$ to produce $y = 2\sin(3(x - \frac{\pi}{4})) + 1$.

EXERCISE 12B.2

1 Sketch the graphs of the following for $0 \leqslant x \leqslant 4\pi$:

a $y = \sin x - 2$ **b** $y = \sin(x - 2)$ **c** $y = \sin(x + 2)$

d $y = \sin x + 2$ **e** $y = \sin(x + \frac{\pi}{4})$ **f** $y = \sin(x - \frac{\pi}{6}) + 1$

GRAPHING PACKAGE

Check your answers using technology.

2 State the period of:

a $y = \sin 5t$ **b** $y = \sin\left(\frac{t}{4}\right)$ **c** $y = \sin(-2t)$

3 Find b in $y = \sin bx$ if $b > 0$ and the period is:

a 3π **b** $\frac{\pi}{10}$ **c** 100π **d** 50

4 State the transformations which map:

a $y = \sin x$ onto $y = \sin x - 1$
b $y = \sin x$ onto $y = \sin(x - \frac{\pi}{4})$
c $y = \sin x$ onto $y = 2\sin x$
d $y = \sin x$ onto $y = \sin 4x$
e $y = \sin x$ onto $y = \frac{1}{2}\sin x$
f $y = \sin x$ onto $y = \sin\left(\frac{x}{4}\right)$
g $y = \sin x$ onto $y = -\sin x$
h $y = \sin x$ onto $y = -3 + \sin(x + 2)$
i $y = \sin x$ onto $y = 2\sin 3x$
j $y = \sin x$ onto $y = \sin(x - \frac{\pi}{3}) + 2$

C MODELLING USING SINE FUNCTIONS

When patterns of variation can be identified and quantified using a formula or equation, predictions may be made about behaviour in the future. Examples of this include tidal movement which can be predicted many months ahead, and the date of a future full moon.

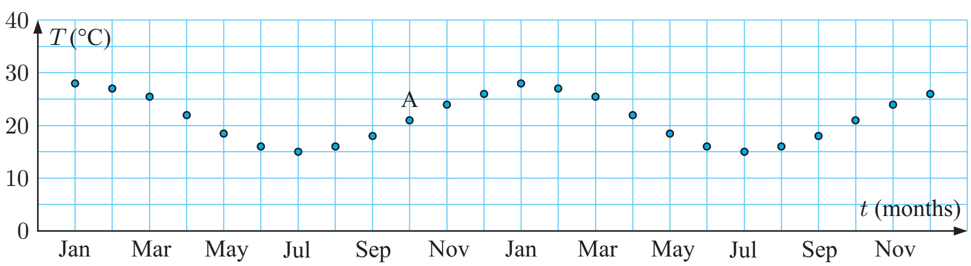
In this section we use sine functions to model periodic biological and physical phenomena.

MEAN MONTHLY TEMPERATURE

Consider again the mean monthly maximum temperature for Cape Town over a 12 month period:

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Temp. T (°C)	28	27	$25\frac{1}{2}$	22	$18\frac{1}{2}$	16	15	16	18	$21\frac{1}{2}$	24	26

The graph over a two year period is shown below:



We attempt to model this data using the general sine function $y = a\sin(b(x-c)) + d$,
or in this case $T = a\sin(b(t-c)) + d$.

The period is 12 months, so $\frac{2\pi}{b} = 12$ and $\therefore\ b = \frac{\pi}{6}$.

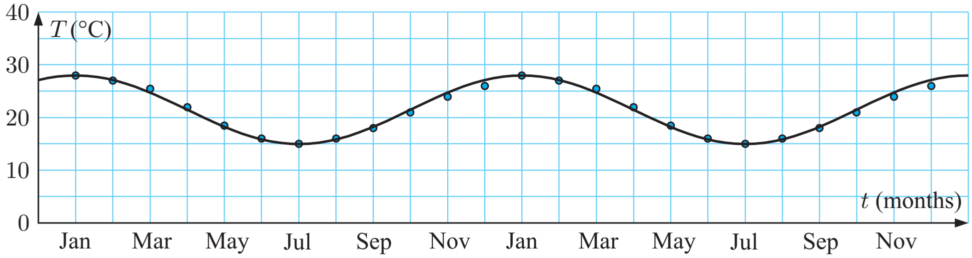
The amplitude $= \frac{\text{max} - \text{min}}{2} \approx \frac{28-15}{2} \approx 6.5$, so $a \approx 6.5$.

The principal axis is midway between the maximum and minimum, so $d \approx \frac{28+15}{2} \approx 21.5$.

So, the model is $T \approx 6.5\sin(\frac{\pi}{6}(t-c)) + 21.5$ for some constant c.

On the original graph, point A lies on the principal axis, and is the first point shown at which we are starting a new period. Since A is at (10, 21.5), $c = 10$.

The model is therefore $T \approx 6.5\sin(\frac{\pi}{6}(t-10)) + 21.5$, and we can superimpose it on the original data as follows.



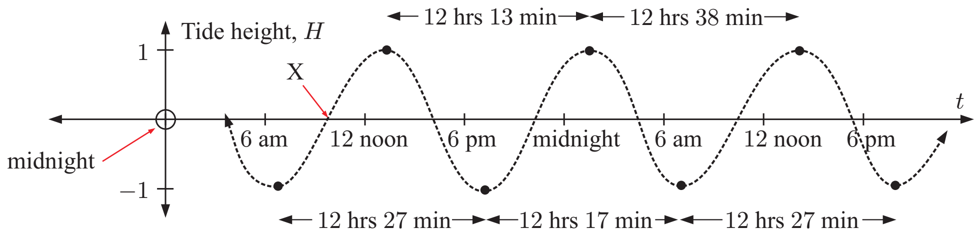
TIDAL MODELS

The tides at Juneau, Alaska were recorded over a two day period. The results are shown in the table opposite:

Day 1	high tide	1:18 pm
	low tide	6:46 am, 7:13 pm
Day 2	high tide	1:31 am, 2:09 pm
	low tide	7:30 am, 7:57 pm

Suppose high tide corresponds to height 1 and low tide to height -1.

Plotting these times with t being the time after midnight before the first low tide, we get:



We attempt to model this periodic data using $H = a\sin(b(t-c)) + d$.

The principal axis is $H = 0$, so $d = 0$.

The amplitude is 1, so $a = 1$.

The graph shows that the 'average' period is about 12 hours 24 min $\approx$ 12.4 hours.

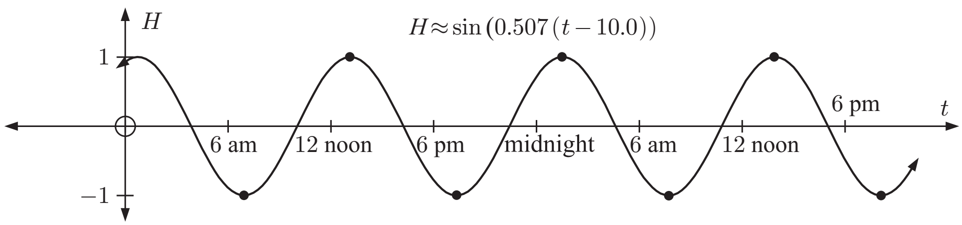
But the period is $\frac{2\pi}{b}$, so $\frac{2\pi}{b} \approx 12.4$ and $\therefore \; b \approx \frac{2\pi}{12.4} \approx 0.507$.

The model is now $H \approx \sin(0.507(t-c))$ for some constant c.

We find point X which is midway between the *first minimum* 6:46 am and the *following maximum* 1:18 pm. Its x-coordinate is $\frac{6.77+13.3}{2} \approx 10.0$, so $c \approx 10.0$.

So, the model is $H \approx \sin(0.507(t-10.0))$.

Below is our original graph of seven plotted points and our model which attempts to fit them.



Use your **graphics calculator** to check this result.
The times must be given in hours after midnight, so
the low tide at 6:46 am is $(6.77, -1)$,
the high tide at 1:18 pm is $(13.3, 1)$, and so on.

GRAPHICS CALCULATOR INSTRUCTIONS

EXERCISE 12C

1 Below is a table which shows the mean monthly maximum temperatures for a city in Greece.

Month	Jan	Feb	Mar	Apr	May	Jun	July	Aug	Sept	Oct	Nov	Dec
Temperature (°C)	15	14	15	18	21	25	27	26	24	20	18	16

a Use a sine function of the form $T \approx a\sin(b(t-c))+d$ to model the data. Find good estimates of the constants a, b, c and d without using technology. Use Jan $\equiv 1$, Feb $\equiv 2$, and so on.

b Use technology to check your answer to **a**. How well does your model fit?

2 The data in the table shows the mean monthly temperatures for Christchurch, New Zealand.

Month	Jan	Feb	Mar	Apr	May	Jun	July	Aug	Sept	Oct	Nov	Dec
Temperature (°C)	15	16	$14\frac{1}{2}$	12	10	$7\frac{1}{2}$	7	$7\frac{1}{2}$	$8\frac{1}{2}$	$10\frac{1}{2}$	$12\frac{1}{2}$	14

a Find a sine model for this data in the form $T \approx a\sin(b(t-c))+d$. Assume Jan $\equiv 1$, Feb $\equiv 2$, and so on. Do not use technology.

b Use technology to check your answer to **a**.

3 Some of the largest tides in the world are observed in Canada's Bay of Fundy. The difference between high and low tides is 14 metres, and the average time difference between high tides is about 12.4 hours.

a Find a sine model for the height of the tide H in terms of the time t.

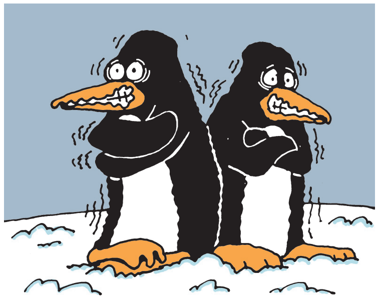
b Sketch the graph of the model over one period.

4 At the Mawson base in Antarctica, the mean monthly temperatures for the last 30 years are:

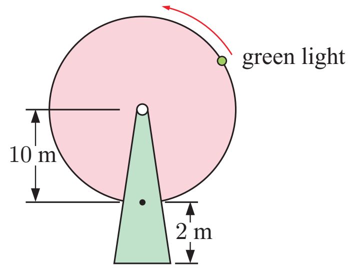
Month	Jan	Feb	Mar	Apr	May	Jun	July	Aug	Sept	Oct	Nov	Dec
Temperature (°C)	0	0	−4	−9	−14	−17	−18	−19	−17	−13	−6	−2

a Find a sine model for this data without using technology. Use Jan $\equiv 1$, Feb $\equiv 2$, and so on.

b How appropriate is the model?



5 Revisit the **Opening Problem** on page **326**. The wheel takes 100 seconds to complete one revolution. Find the sine model which gives the height of the light above the ground at any point in time. Assume that at time $t = 0$, the light is at its lowest point.

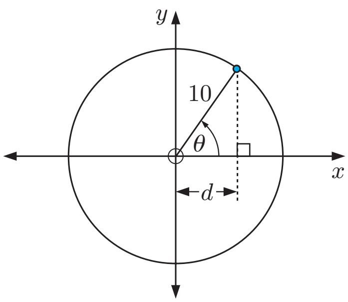


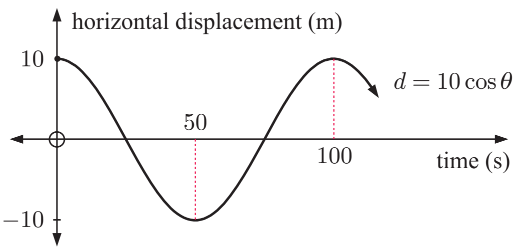
D THE COSINE FUNCTION

We return to the Ferris wheel and now view the movement of the green light from above.

Now $\cos\theta = \dfrac{d}{10}$ so $d = 10\cos\theta$

The graph being generated over time is therefore a **cosine function**.





DEMO

The cosine curve $y = \cos x$, like the sine curve $y = \sin x$, has a **period** of 2π, an **amplitude** of 1, and its **range** is $-1 \leqslant y \leqslant 1$.

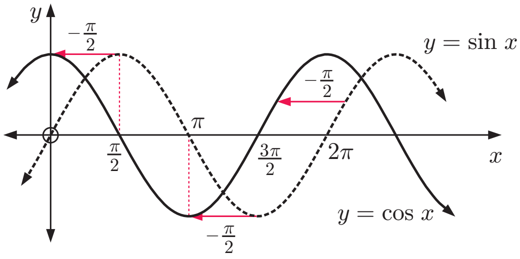
Use your graphics calculator or graphing package to check these features.

GRAPHICS CALCULATOR INSTRUCTIONS

GRAPHING PACKAGE

Now view the relationship between the sine and cosine functions.

You should observe that $y = \cos x$ and $y = \sin x$ are identical in shape, but the cosine function is $\frac{\pi}{2}$ units left of the sine function under a horizontal translation.



This suggests that $\mathbf{\cos x = \sin\left(x + \frac{\pi}{2}\right)}$.

Use your graphing package or graphics calculator to check this by graphing $y = \cos x$ and $y = \sin\left(x + \frac{\pi}{2}\right)$ on the same set of axes.

GRAPHING PACKAGE

THE GENERAL COSINE FUNCTION

The **general cosine function** is $y = a\cos(b(x-c)) + d$ where $a \neq 0$, $b > 0$.

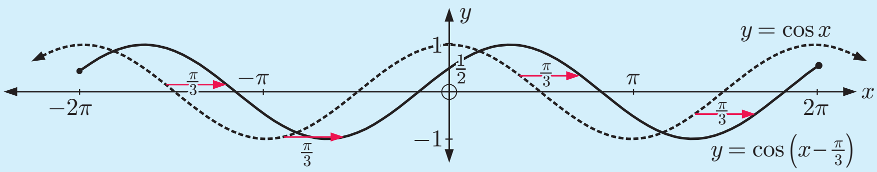
Since the cosine function is a horizontal translation of the sine function, the constants a, b, c, and d have the same effects as for the general sine function. Click on the icon to check this.

DYNAMIC COSINE FUNCTION

Example 4 Self Tutor

On the same set of axes graph $y = \cos x$ and $y = \cos\left(x - \frac{\pi}{3}\right)$ for $-2\pi \leqslant x \leqslant 2\pi$.

$y = \cos\left(x - \frac{\pi}{3}\right)$ is a horizontal translation of $y = \cos x$ through $\frac{\pi}{3}$ units to the right.



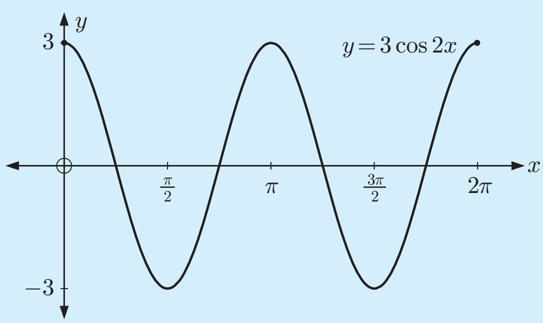
Example 5 Self Tutor

Without using technology, sketch the graph of $y = 3\cos 2x$ for $0 \leqslant x \leqslant 2\pi$.

Notice that $a = 3$, so the amplitude is $|3| = 3$.

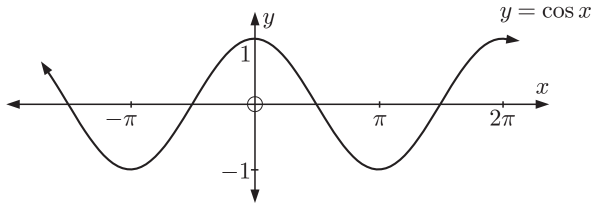
$b = 2$, so the period is $\frac{2\pi}{b} = \frac{2\pi}{2} = \pi$.

To obtain this from $y = \cos x$, we have a vertical stretch with scale factor 3 followed by a horizontal stretch with scale factor $\frac{1}{2}$, as the period has been halved.



EXERCISE 12D

1 Given the graph of $y = \cos x$, sketch the graphs of:



a $y = \cos x + 2$ **b** $y = \cos x - 1$ **c** $y = \cos\left(x - \frac{\pi}{4}\right)$

d $y = \cos\left(x + \frac{\pi}{6}\right)$ **e** $y = \frac{2}{3}\cos x$ **f** $y = \frac{3}{2}\cos x$

g $y = -\cos x$ **h** $y = \cos\left(x - \frac{\pi}{6}\right) + 1$ **i** $y = \cos\left(x + \frac{\pi}{4}\right) - 1$

j $y = \cos 2x$ **k** $y = \cos\left(\frac{x}{2}\right)$ **l** $y = 3\cos 2x$

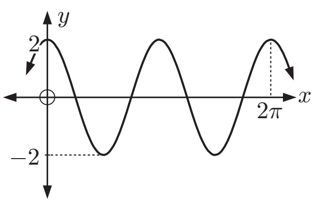
2 Without graphing them, state the periods of:

a $y = \cos 3x$ **b** $y = \cos\left(\frac{x}{3}\right)$ **c** $y = \cos\left(\frac{\pi}{50}x\right)$

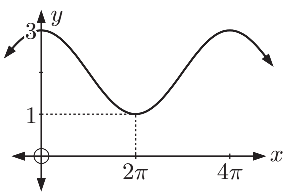
3 The general cosine function is $y = a\cos(b(x - c)) + d$.
State the geometrical significance of a, b, c, and d.

4 Find the cosine function shown in the graph:

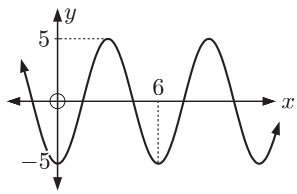
a



b



c

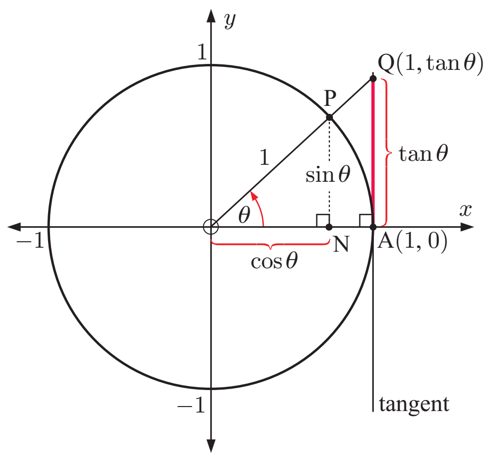


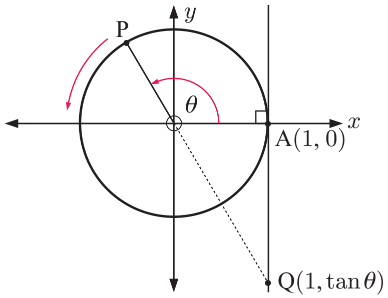
E THE TANGENT FUNCTION

We have seen that if $P(\cos\theta, \sin\theta)$ is a point which is free to move around the unit circle, and if [OP] is extended to meet the tangent at $A(1, 0)$, the intersection between these lines occurs at $Q(1, \tan\theta)$.

This enables us to define the **tangent function**

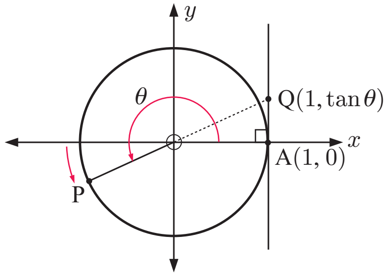
$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$



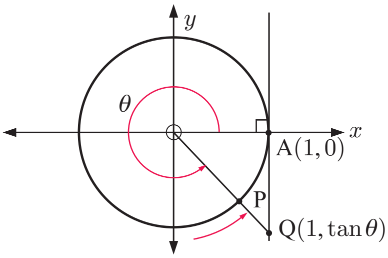


For θ in quadrant 2, $\sin\theta$ is positive and $\cos\theta$ is negative and so $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ is negative.

As before, [OP] is extended to meet the tangent at A at Q$(1, \tan\theta)$.



For θ in quadrant 3, $\sin\theta$ and $\cos\theta$ are both negative and so $\tan\theta$ is positive. This is clearly demonstrated as Q is back above the x-axis.



For θ in quadrant 4, $\sin\theta$ is negative and $\cos\theta$ is positive. $\tan\theta$ is again negative.

DISCUSSION

What happens to $\tan\theta$ when P is at $(0, 1)$ and $(0, -1)$?

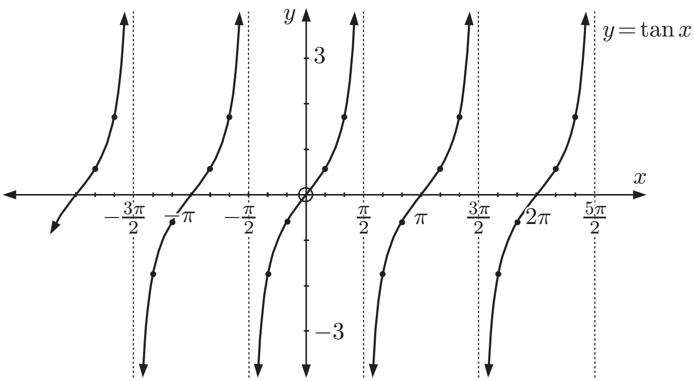
THE GRAPH OF $y = \tan x$

Since $\tan x = \dfrac{\sin x}{\cos x}$, $\tan x$ will be undefined whenever $\cos x = 0$.

The zeros of the function $y = \cos x$ correspond to vertical asymptotes of the function $y = \tan x$.

The graph of $y = \tan x$ is

DEMO



We observe that $y = \tan x$ has:
- **period** of π
- **range** $y \in \mathbb{R}$
- **vertical asymptotes** $x = \frac{\pi}{2} + k\pi$ for all $k \in \mathbb{Z}$.

Click on the icon to explore how the tangent function is produced from the unit circle.

TANGENT FUNCTION

DISCUSSION

- Discuss how to find the x-intercepts of $y = \tan x$.
- What must $\tan(x - \pi)$ simplify to?
- How many solutions does the equation $\tan x = 2$ have?

THE GENERAL TANGENT FUNCTION

The **general tangent function** is $y = a\tan(b(x - c)) + d$, $a \neq 0$, $b > 0$.
- The **principal axis** is $y = d$.
- The **period** of this function is $\dfrac{\pi}{b}$.
- The **amplitude** of this function is undefined.
- There are infinitely many vertical asymptotes.

Click on the icon to explore the properties of this function.

DYNAMIC TANGENT FUNCTION

Example 6 Self Tutor

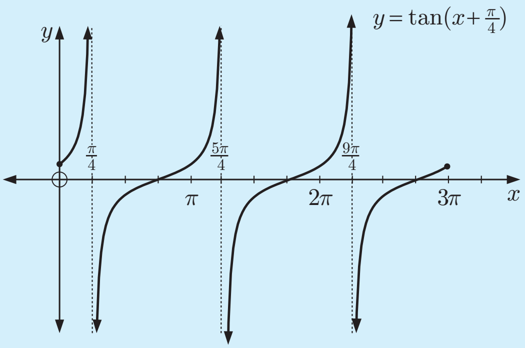
Without using technology, sketch the graph of $y = \tan(x + \frac{\pi}{4})$ for $0 \leqslant x \leqslant 3\pi$.

$y = \tan(x + \frac{\pi}{4})$ is a horizontal translation of $y = \tan x$ through $-\frac{\pi}{4}$

$y = \tan x$ has vertical asymptotes $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$, and $x = \frac{5\pi}{2}$

Its x-axis intercepts are 0, π, 2π, and 3π.

$\therefore$ $y = \tan(x + \frac{\pi}{4})$ has vertical asymptotes $x = \frac{\pi}{4}$, $x = \frac{5\pi}{4}$, $x = \frac{9\pi}{4}$, and x-intercepts $\frac{3\pi}{4}$, $\frac{7\pi}{4}$, and $\frac{11\pi}{4}$.



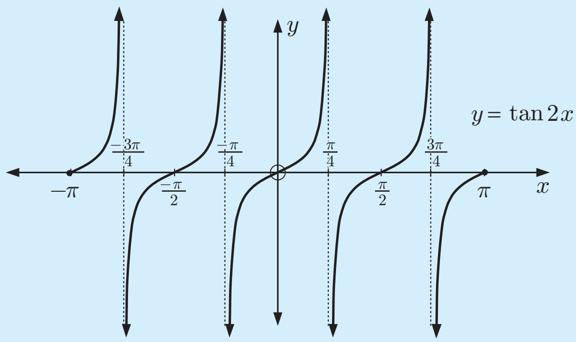
Example 7 — Self Tutor

Without using technology, sketch the graph of $y = \tan 2x$ for $-\pi \leqslant x \leqslant \pi$.

$y = \tan 2x$ is a horizontal stretch of $y = \tan x$ with scale factor $\frac{1}{2}$.

Since $b = 2$, the period is $\frac{\pi}{2}$.

The vertical asymptotes are $x = \pm\frac{\pi}{4}$, $x = \pm\frac{3\pi}{4}$ and the x-axis intercepts are at $0, \pm\frac{\pi}{2}, \pm\pi$.



EXERCISE 12E

1 **a** *Sketch* the following functions for $0 \leqslant x \leqslant 3\pi$:

  i $y = \tan(x - \frac{\pi}{2})$ **ii** $y = -\tan x$ **iii** $y = \tan 3x$

b Use technology to check your answers to **a**.
Look in particular for asymptotes and the x-intercepts.

GRAPHING PACKAGE

2 Describe the transformation(s) which moves the first curve to the second curve:

  a $y = \tan x$ to $y = \tan(x - 1) + 2$ **b** $y = \tan x$ to $y = -\tan x$

  c $y = \tan x$ to $y = 2\tan\left(\frac{x}{2}\right)$

3 State the period of:

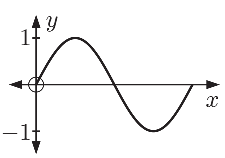
  a $y = \tan x$ **b** $y = \tan 3x$ **c** $y = \tan nx$, $n \neq 0$

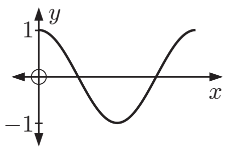
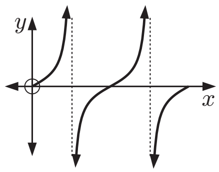
GRAPHING PACKAGE

F GENERAL TRIGONOMETRIC FUNCTIONS

In the previous sections we have explored properties of the sine, cosine, and tangent functions, and observed how they can be transformed into more general trigonometric functions.

The following tables summarise our observations:

FEATURES OF CIRCULAR FUNCTIONS					
Function	*Sketch for* $0 \leqslant x \leqslant 2\pi$	*Period*	*Amplitude*	*Domain*	*Range*
$y = \sin x$		2π	1	$x \in \mathbb{R}$	$-1 \leqslant y \leqslant 1$

Function	*Sketch for* $0 \leqslant x \leqslant 2\pi$	*Period*	*Amplitude*	*Domain*	*Range*
$y = \cos x$		2π	1	$x \in \mathbb{R}$	$-1 \leqslant y \leqslant 1$
$y = \tan x$		π	undefined	$x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, ...$	$y \in \mathbb{R}$

GENERAL TRIGONOMETRIC FUNCTIONS				
General function	a affects vertical stretch	$b > 0$ affects horizontal stretch	c affects horizontal translation	d affects vertical translation
$y = a\sin(b(x-c)) + d$ $y = a\cos(b(x-c)) + d$	amplitude $= \lvert a \rvert$	period $= \frac{2\pi}{b}$	• $c > 0$ moves the graph right • $c < 0$ moves the graph left	• $d > 0$ moves the graph up • $d < 0$ moves the graph down • principal axis is $y = d$
$y = a\tan(b(x-c)) + d$	amplitude undefined	period $= \frac{\pi}{b}$		

EXERCISE 12F

1 State the amplitude, where appropriate, of:

a $y = \sin 4x$ **b** $y = 2\tan(\frac{x}{2})$ **c** $y = -\cos(3(x - \frac{\pi}{4}))$

2 State the period of:

a $y = -\tan x$ **b** $y = \cos(\frac{x}{3}) - 1$ **c** $y = \sin(2(x - \frac{\pi}{4}))$

3 Find b given:

a $y = \sin bx$ has period 2π **b** $y = \cos bx$ has period $\frac{2\pi}{3}$

c $y = \tan bx$ has period $\frac{\pi}{2}$ **d** $y = \sin bx$ has period 4

4 Sketch the graphs of these functions for $0 \leqslant x \leqslant 2\pi$:

a $y = \frac{2}{3}\cos x$ **b** $y = \sin x + 1$ **c** $y = \tan(x + \frac{\pi}{2})$

d $y = 3\cos 2x$ **e** $y = \sin(x + \frac{\pi}{4}) - 1$ **f** $y = \tan x - 2$

5 State the maximum and minimum values, where appropriate, of:

a $y = -\sin 5x$ **b** $y = 3\cos x$ **c** $y = 2\tan x$

d $y = -\cos 2x + 3$ **e** $y = 1 + 2\sin x$ **f** $y = \sin(x - \frac{\pi}{2}) - 3$

6 State the transformation(s) which map(s):

a $y = \sin x$ onto $y = \frac{1}{2}\sin x$

b $y = \cos x$ onto $y = \cos(\frac{x}{4})$

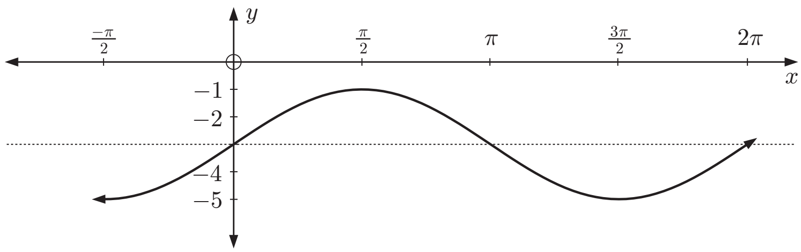
c $y = \sin x$ onto $y = -\sin x$

d $y = \cos x$ onto $y = \cos x - 2$

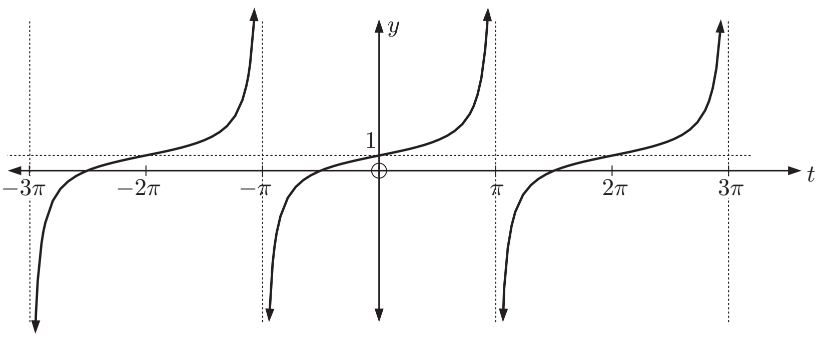
e $y = \tan x$ onto $y = \tan(x + \frac{\pi}{4})$

f $y = \sin x$ onto $y = \sin(-x)$

7 Find m and n given the following graph is of the function $y = m\sin x + n$.



8 Find p and q given the following graph is of the function $y = \tan pt + q$.



ACTIVITY

Click on the icon to run a card game for trigonometric functions.

G RECIPROCAL TRIGONOMETRIC FUNCTIONS

We define the reciprocal trigonometric functions cosec x, secant x, and cotangent x as:

$$\csc x = \frac{1}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \text{and} \quad \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$

Using these definitions we can derive the identities:

$$\tan^2 x + 1 = \sec^2 x \quad \text{and} \quad 1 + \cot^2 x = \csc^2 x$$

Proof: Using $\sin^2 x + \cos^2 x = 1$,

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \qquad \{\text{dividing each term by } \cos^2 x\}$$

$$\therefore \tan^2 x + 1 = \sec^2 x$$

Also using $\sin^2 x + \cos^2 x = 1$,

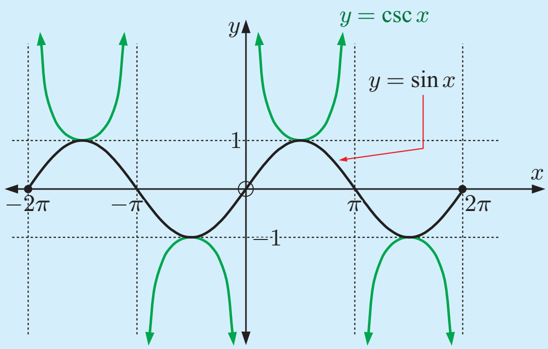
$$\frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x} \qquad \{\text{dividing each term by } \sin^2 x\}$$

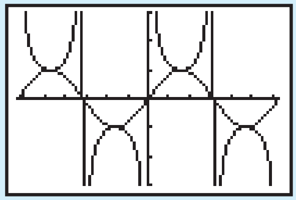
$$\therefore \quad 1 + \cot^2 x = \csc^2 x$$

Example 8 **Self Tutor**

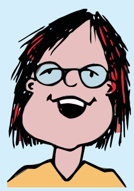
Using the graph of $y = \sin x$, sketch the graph of $y = \dfrac{1}{\sin x} = \csc x$ for $x \in [-2\pi, 2\pi]$.

Check your answer using technology.





Use the techniques for graphing $\dfrac{1}{f(x)}$ from **Chapter 5**.



EXERCISE 12G

1 Without using a calculator, find:

a $\csc\left(\frac{\pi}{3}\right)$ **b** $\cot\left(\frac{2\pi}{3}\right)$ **c** $\sec\left(\frac{5\pi}{6}\right)$ **d** $\cot(\pi)$

2 Without using a calculator, find $\csc x$, $\sec x$, and $\cot x$ given:

a $\sin x = \frac{3}{5}$, $0 \leqslant x \leqslant \frac{\pi}{2}$ **b** $\cos x = \frac{2}{3}$, $\frac{3\pi}{2} < x < 2\pi$

3 Find the other *five* trigonometric ratios if:

a $\cos x = \frac{3}{4}$ and $\frac{3\pi}{2} < x < 2\pi$ **b** $\sin x = -\frac{2}{3}$ and $\pi < x < \frac{3\pi}{2}$

c $\sec x = 2\frac{1}{2}$ and $0 < x < \frac{\pi}{2}$ **d** $\csc x = 2$ and $\frac{\pi}{2} < x < \pi$

e $\tan\beta = \frac{1}{2}$ and $\pi < \beta < \frac{3\pi}{2}$ **f** $\cot\theta = \frac{4}{3}$ and $\pi < \theta < \frac{3\pi}{2}$

4 Simplify:

a $\tan x \cot x$ **b** $\sin x \csc x$ **c** $\csc x \cot x$

d $\sin x \cot x$ **e** $\dfrac{\cot x}{\csc x}$ **f** $\dfrac{2\sin x \cot x + 3\cos x}{\cot x}$

5 Using the graph of $y = \cos x$, sketch the graph of $y = \dfrac{1}{\cos x} = \sec x$ for $x \in [-2\pi, 2\pi]$. Check your answer using technology.

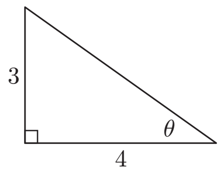
6 Using the graph of $y = \tan x$, sketch the graph of $y = \dfrac{1}{\tan x} = \cot x$ for $x \in [-2\pi, 2\pi]$.

7 Use technology to sketch $y = \sec x$ and $y = \csc\left(x + \frac{\pi}{2}\right)$ on the same set of axes for $x \in [-2\pi, 2\pi]$. Explain your answer.

H INVERSE TRIGONOMETRIC FUNCTIONS

In many problems we need to know what angle results in a particular trigonometric ratio. We have already seen this for right angled triangle problems and the cosine and sine rules.

For example:



$\tan\theta = \dfrac{\text{OPP}}{\text{ADJ}} = \dfrac{3}{4}$

$\therefore \quad \theta = \tan^{-1}(\frac{3}{4})$

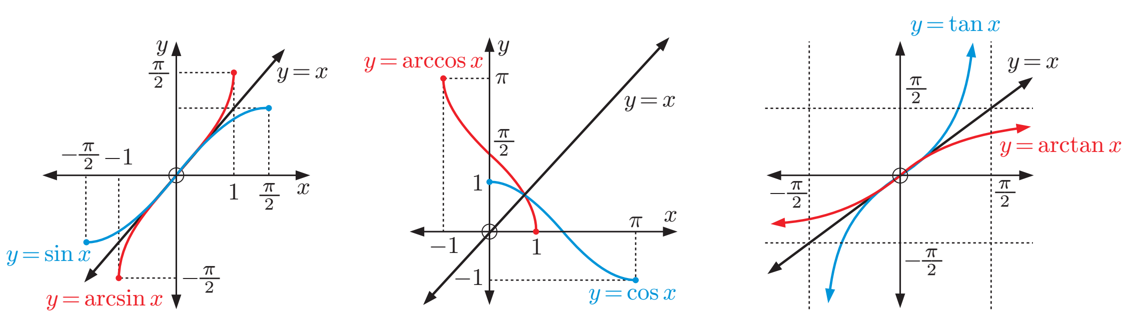
$\therefore \quad \theta \approx 36.9°$

Rather than writing the inverse trigonometric functions as $\sin^{-1} x$, $\cos^{-1} x$, and $\tan^{-1} x$, which can be confused with reciprocals, mathematicians more formally refer to these functions as arcsine x, arccosine x, and arctangent x.

Since $\sin x$, $\cos x$, and $\tan x$ are all many-to-one functions, their domains must be restricted in order for them to have inverse functions. The inverse functions are therefore defined as follows:

Function	*Definition*	*Range*
$y = \arcsin x$	$x = \sin y, \quad -1 \leqslant x \leqslant 1$	$-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$
$y = \arccos x$	$x = \cos y, \quad -1 \leqslant x \leqslant 1$	$0 \leqslant y \leqslant \pi$
$y = \arctan x$	$x = \tan y, \quad x \in \mathbb{R}$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$

The graphs of these functions are illustrated below as the inverse functions of $\sin x$, $\cos x$, and $\tan x$ on restricted domains which are the ranges of $\arcsin x$, $\arccos x$, and $\arctan x$ respectively.



EXERCISE 12H

1 Copy and complete the table below, giving the restricted domain and range for each trigonometric function so that its inverse function exists:

Function	*Restricted domain*	*Restricted range*	*Inverse function*	*Domain*	*Range*
$y = \sin x$			$y = \arcsin x$		
$y = \cos x$			$y = \arccos x$		
$y = \tan x$			$y = \arctan x$		

2 Find, giving your answer in radians:

a $\arccos(1)$ **b** $\arcsin(-1)$ **c** $\arctan(1)$ **d** $\arctan(-1)$

e $\arcsin\left(\frac{1}{2}\right)$ **f** $\arccos\left(\frac{-\sqrt{3}}{2}\right)$ **g** $\arctan\left(\sqrt{3}\right)$ **h** $\arccos\left(-\frac{1}{\sqrt{2}}\right)$

i $\arctan\left(-\frac{1}{\sqrt{3}}\right)$ **j** $\sin^{-1}(-0.767)$ **k** $\cos^{-1}(0.327)$ **l** $\tan^{-1}(-50)$

3 Find the invariant point for the inverse transformation from:

a $y = \sin x$ to $y = \arcsin x$ **b** $y = \tan x$ to $y = \arctan x$

c $y = \cos x$ to $y = \arccos x$.

4 **a** State the equations of the asymptotes of $y = \arctan x$.

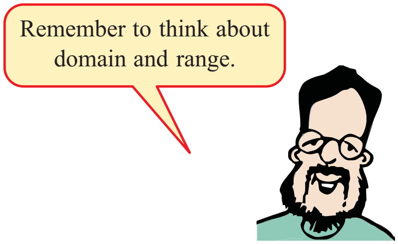
b Do the functions $y = \arcsin x$ and $y = \arccos x$ have vertical asymptotes? Explain your answer.

5 Simplify:

a $\arcsin(\sin\frac{\pi}{3})$ **b** $\arccos(\cos(-\frac{\pi}{6}))$

c $\tan(\arctan(0.3))$ **d** $\cos(\arccos(-\frac{1}{2}))$

e $\arctan(\tan\pi)$ **f** $\arcsin(\sin\frac{4\pi}{3})$



INVESTIGATION 4

Carl Friedrich Gauss used his **Gaussian hypergeometric series** to analyse the continued fraction:

$$\cfrac{x}{1 + \cfrac{(1x)^2}{3 + \cfrac{(2x)^2}{5 + \cfrac{(3x)^2}{7 + \cfrac{(4x)^2}{9 + ...}}}}}$$

What to do:

1 Evaluate the fraction with $x = 1$ for as many levels as necessary for the answer to be accurate to 5 decimal places. You may wish to use a spreadsheet.

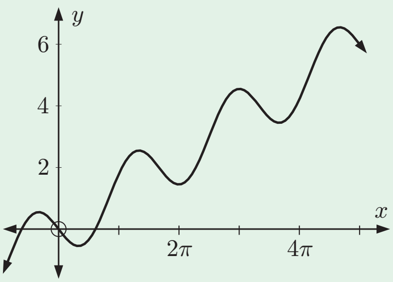
2 Compare your result with $\arctan 1$.

3 Compare the continued fraction and $\arctan x$ for another value of x of your choosing.

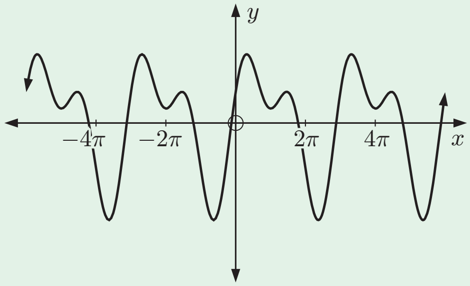
REVIEW SET 12A — NON-CALCULATOR

1 Which of the following graphs display periodic behaviour?

a



b



2 Draw the graph of $y = 4\sin x$ for $0 \leqslant x \leqslant 2\pi$.

3 State the minimum and maximum values of:

a $1 + \sin x$ **b** $-2\cos 3x$

4 State the period of:

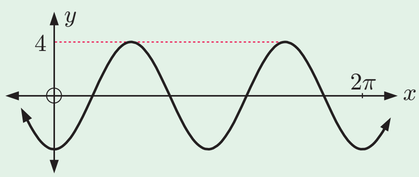
a $y = 4\sin(\frac{x}{5})$ **b** $y = -2\cos(4x)$ **c** $y = 4\cos(\frac{x}{2}) + 4$ **d** $y = \frac{1}{2}\tan(3x)$

5 Complete the table:

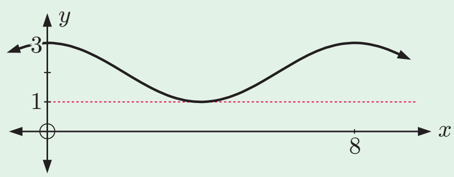
Function	*Period*	*Amplitude*	*Domain*	*Range*
$y = -3\sin(\frac{x}{4}) + 1$				
$y = \tan 2x$				
$y = 3\cos \pi x$				

6 Find the cosine function represented in each of the following graphs:

a



b



7 State the transformations which map:

a $y = \sin x$ onto $y = 3\sin(2x)$ **b** $y = \cos x$ onto $y = \cos\left(x - \frac{\pi}{3}\right) - 1$

8 Find the remaining five trigonometric ratios from sin, cos, tan, csc, sec, and cot, if:

a $\cos x = \frac{1}{3}$ and $0 < x < \pi$ **b** $\tan x = \frac{4}{5}$ and $\pi < x < 2\pi$.

9 Simplify:

a $\arctan(\tan(-0.5))$ **b** $\arcsin(\sin(-\frac{\pi}{6}))$ **c** $\arccos(\cos 2\pi)$

REVIEW SET 12B — CALCULATOR

1 For each set of data below, draw a scatter diagram and state if the data exhibits approximately periodic behaviour.

a

x	0	1	2	3	4	5	6	7	8	9	10	11	12
y	2.7	0.8	-1.7	-3	-2.1	0.3	2.5	2.9	1.3	-1.3	-2.9	-2.5	-0.3

b

x	0	1	2	3	4	5	6	7	8	9
y	5	3.5	6	-1.5	4	-2.5	-0.8	0.9	2.6	4.3

2 Draw the graph of $y = \sin 3x$ for $0 \leqslant x \leqslant 2\pi$.

3 State the period of: **a** $y = 4\sin(\frac{x}{3})$ **b** $y = -2\tan 4x$

4 Draw the graph of $y = 0.6\cos(2.3x)$ for $0 \leqslant x \leqslant 5$.

5 A robot on Mars records the temperature every Mars day. A summary series, showing every one hundredth Mars day, is shown in the table below.

Number of Mars days	0	100	200	300	400	500	600	700	800	900	1000	1100	1200	1300
Temp. (°C)	-43	-15	-5	-21	-59	-79	-68	-50	-27	-8	-15	-70	-78	-68

a Find the maximum and minimum temperatures recorded by the robot.

b Find a sine model for the temperature T in terms of the number of Mars days n.

c Use this information to estimate the length of a Mars year.

6 State the minimum and maximum values of:

a $y = 5\sin x - 3$ **b** $y = \frac{1}{3}\cos x + 1$

7 State the transformations which map:

a $y = \tan x$ onto $y = -\tan(2x)$ **b** $y = \sin x$ onto $y = 2\sin\left(\frac{x}{2} - \frac{\pi}{4}\right) + \frac{1}{2}$

8 **a** Sketch the graphs of $y = \sec x$ and $y = \csc x$ on the same set of axes for $-2\pi \leqslant x \leqslant 2\pi$.

b State a transformation which maps $y = \sec x$ onto $y = \csc x$ for all $x \in \mathbb{R}$.

9 **a** Sketch the graphs of $y = \arcsin x$ and $y = \arccos x$ on the same set of axes.

b State the domain and range of each function.

c State the transformations which map $y = \arcsin x$ onto $y = \arccos x$.

REVIEW SET 12C

1 Find b given that the function $y = \sin bx$, $b > 0$ has period:

a 6π **b** $\frac{\pi}{12}$ **c** 9

2 **a** Without using technology, draw the graph of $f(x) = \sin(x - \frac{\pi}{3}) + 2$ for $0 \leqslant x \leqslant 2\pi$.

b For what values of k will $f(x) = k$ have solutions?

3 Consider the graph alongside.

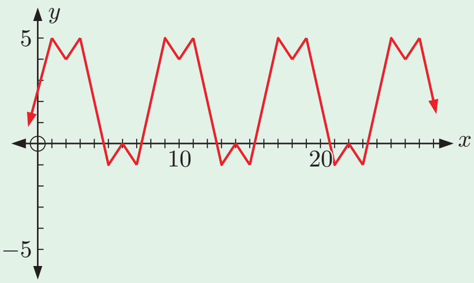
a Explain why this graph shows periodic behaviour.

b State:

i the period

ii the maximum value

iii the minimum value



4 On the same set of axes, for the domain $0 \leqslant x \leqslant 2\pi$, sketch:

a $y = \cos x$ and $y = \cos x - 3$

b $y = \cos x$ and $y = \cos(x - \frac{\pi}{4})$

c $y = \cos x$ and $y = 3\cos 2x$

d $y = \cos x$ and $y = 2\cos(x - \frac{\pi}{3}) + 3$

5 The table below gives the mean monthly maximum temperature for Perth Airport in Western Australia.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Temp. (°C)	31.5	31.8	29.5	25.4	21.5	18.8	17.7	18.3	20.1	22.4	25.5	28.8

a A sine function of the form $T \approx a\sin(b(t - c)) + d$ is used to model the data. Find good estimates of the constants a, b, c, and d without using technology. Use Jan $\equiv 1$, Feb $\equiv 2$, and so on.

b Check your answer to **a** using technology. How well does your model fit?

6 State the transformations which map:

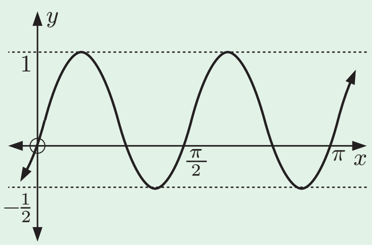
a $y = \cos x$ onto $y = \cos(x - \frac{\pi}{3}) + 1$

b $y = \tan x$ onto $y = -2\tan x$

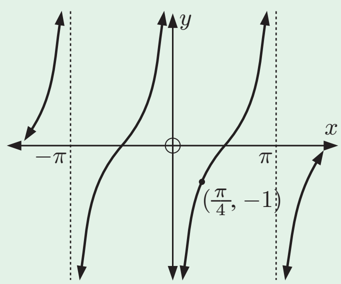
c $y = \sin x$ onto $y = \sin(3x)$

7 Find the function represented in each of the following graphs:

a



b



8 Simplify:

a $\csc x \tan x$

b $\dfrac{\tan x}{\sec x}$

c $\sec x - \tan x \sin x$

9 **a** For what restricted domain of $y = \tan x$, is $y = \arctan x$ the inverse function?

b Sketch $y = \tan x$ for this domain, and $y = \arctan x$, on the same set of axes.